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ON THE BINARY QUADRATIC EQUATION $ax^2 - (a+1)y^2 = a$

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ABSTRACT

The binary quadratic equation $ax^2 - (a+1)y^2 = a$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

KEYWORDS: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation. 2010 Mathematics subject classification: 11D09.

INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non-homogeneous) are rich in variety.[1-5] In[6-12] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of another interesting binary quadratic equation given by $ax^2 - (a+1)y^2 = a$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

METHOD OF ANALYSIS

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$ax^2-(a+1)y^2=a \quad (1)$$

Substituting the linear transformations

$$x=X\pm(a+1)T \quad , \quad y=X\pm aT \quad (2)$$

in (1) , we have

$$X^2=a(a+1)T^2-a \quad (3)$$

The least positive integer solution is

$$T_0=1, X_0=a \quad (4)$$

Now , to find the other solutions of (3) , consider the pellian equation

$$X^2=a(a+1)T^2+1 \quad (5)$$

whose fundamental solution is

$$\left(\tilde{T}_0,\tilde{X}_0\right)=(2,2a+1)$$

The other solutions of (6) can be derived from the relations

$$\tilde{X}_n=\frac{f_n}{2}$$

$$\tilde{T}_n=\frac{g_n}{2\sqrt{a^2+a}}$$

where

$$f_n=\left(2a+1+2\sqrt{a^2+a}\right)^{n+1}+\left(2a+1-2\sqrt{a^2+a}\right)^{n+1}$$

$$g_n=\left(2a+1+2\sqrt{a^2+a}\right)^{n+1}-\left(2a+1-2\sqrt{a^2+a}\right)^{n+1}$$

Applying the Brahmagupta lemma between (T_0, X_0) and $\left(\tilde{T}_n,\tilde{X}_n\right)$, the other solutions of (3) can be obtained from the relation

$$\begin{aligned} T_{n+1}&=\frac{1}{2}f_n+\frac{a}{2\sqrt{a^2+a}}g_n \\ X_{n+1}&=\frac{a}{2}f_n+\frac{\sqrt{a^2+a}}{2}g_n \end{aligned} \quad (6)$$

By substituting equation (6) in (2), the non-zero distinct integer solutions of (1) are obtained as follows

$$x_{n+1}=\frac{-1}{2}f_n \ , \left[\frac{2a+1}{2}f_n+\sqrt{a^2+a}\,g_n\right]$$

$$y_{n+1}=\frac{\sqrt{a^2+a}}{2(a+1)}g_n \ , \left[a\,f_n+\frac{(2a+1)\sqrt{a^2+a}}{2(a+1)}g_n\right]$$

The recurrence relations for x_{n+1}, y_{n+1} are respectively

$$x_{n+3}-(4a+2)x_{n+2}+x_{n+1}=0$$
$$y_{n+3}-(4a+2)y_{n+2}+y_{n+1}=0.$$

From the above solutions we obtain some interesting relations, which are presented below:

1. Relations among the solutions:

- $2x_{n+3}=(8a+4)x_{n+2}-2x_{n+1}$
- $2(2a+1)x_{n+2}=x_{n+1}+2x_{n+3}$
- $x_{n+2}=x_{n+1}-2(a+1)y_{n+2}$
- $2(a+1)y_{n+2}=x_{n+1}-(2a+1)x_{n+2}$
- $(8a^2+8a+1)x_{n+2}=(2a+1)x_{n+1}-2(a+1)y_{n+3}$
- $(2a+1)y_{n+1}=y_{n+2}+2ax_{n+1}$
- $4(a+1)(2a+1)y_{n+1}=(8a^2+8a+1)x_{n+1}-2x_{n+3}$
- $(8a^2+8a+1)y_{n+2}=2ax_{n+1}+(2a+1)y_{n+3}$
- $4(a+1)y_{n+2}=x_{n+1}-2x_{n+3}$
- $2(a+1)y_{n+3}=(2a+1)x_{n+1}-(8a^2+8a+1)x_{n+2}$
- $2(a+1)y_{n+1}=(2a+1)x_{n+1}-x_{n+2}$
- $x_{n+3}=x_{n+1}-4(a+1)y_{n+2}$
- $4(a+1)(2a+1)y_{n+3}=x_{n+1}-2(8a^2+8a+1)x_{n+3}$
- $(8a^2+8a+1)y_{n+1}=4a(2a+1)x_{n+1}+y_{n+3}$
- $(2a+1)y_{n+3}=(8a^2+8a+1)y_{n+2}-2ax_{n+1}$
- $(8a^2+8a+1)x_{n+3}=x_{n+1}-4(a+1)(2a+1)y_{n+3}$
- $2ax_{n+1}=(2a+1)y_{n+1}-y_{n+2}$
- $x_{n+1}=2(2a+1)x_{n+2}-x_{n+3}$
- $2ax_{n+2}=y_{n+1}-(2a+1)y_{n+2}$
- $2(4a^2+5a+1)(8a^2+8a+1)y_{n+1}=(8a^2+8a+1)x_{n+2}-(2a+1)x_{n+3}$
- $2y_{n+3}=4(2a+1)y_{n+2}-2y_{n+1}$
- $2ax_{n+3}=(2a+1)y_{n+1}-(8a^2+8a+1)y_{n+2}$
- $$\begin{aligned}(8a^2+8a+1)\{(8a^2+10a+2)y_{n+2}+(64a^4+112a^3+56a^2+6a-1)x_{n+2}\}\\=(128a^5+288a^4+208a^3+48a^2-1)x_{n+3}\end{aligned}$$

OBSERVATIONS

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented below.

Hyperbolas

- $4(a^2+a)[x_{n+1}^2-1]-[(2a+1)x_{n+1}-x_{n+2}]^2=0.$
- $16(2a+1)^2(a^2+a)[x_{n+1}^2-1]-[(8a^2+8a+1)x_{n+1}-2x_{n+3}]^2=0.$
- $4(2a+1)^2(a^2+a)[x_{n+1}^2-1]-(a+1)^2[2y_{n+2}+4ax_{n+1}]^2=0.$
- $4(8a^2+8a+1)^2(a^2+a)[x_{n+1}^2-1]-(a+1)^2[8a(2a+1)x_{n+1}+2y_{n+3}]^2=0.$
- $(4a+1)^2(a^2+a)(8a^2+8a+1)^2\{[2x_{n+3}-4(2a+1)x_{n+2}]^2-2^2\}-[(8a^2+8a+1)x_{n+2}-(2a+1)x_{n+3}]^2=0.$
- $[y_{n+2}-(2a+1)y_{n+1}]^2-4a(a+1)y_{n+1}^2=4a^2.$
- $[y_{n+3}-(8a^2+8a+1)y_{n+1}]^2-16a(2a+1)[(a+1)y_{n+1}^2+a]=0.$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented below.

Parabolas

- $2a(a+1)[x_{2n+2}+1]+[(2a+1)x_{n+1}-x_{n+2}]=0.$
- $[y_{2n+3}-(2a+1)y_{2n+2}+2a]-4(a+1)y_{n+1}^2=4a.$
- $8a(a+1)(2a+1)^2[1+x_{2n+2}]+[(8a^2+8a+1)x_{n+1}-2x_{n+3}]^2=0.$
- $2a(a+1)(4a+1)^2(8a^2+8a+1)^2[x_{2n+4}-2(2a+1)x_{2n+3}-1]-[(8a^2+8a+1)x_{n+2}-(2a+1)x_{n+3}]^2=0.$

  $2a(a+1)(8a^2+8a+1)^2[1+x_{2n+2}]+(a+1)^2[8a(2a+1)x_{n+1}+2y_{n+3}]^2=0.$
- $2a(a+1)(2a+1)^2[1+x_{2n+2}]+(a+1)^2[2y_{n+2}+4ax_{n+1}]^2=0.$

Generation of Solution

If $x_1=x_0+h$ and $y_1=h-y_0$ is any solution of (1) and we have the following x_0, y_0 also satisfies (1).

Let $x_1=x_0+h$, $y_1=h-y_0$ and $h\neq 0$ (7)

Substituting (7) in (1) and performing a few calculations , we obtain

$$h=2ax_0+2(a+1)y_0$$

and then

$$x_1 = (2a+1)x_0 + (2a+2)y_0$$
$$y_1 = 2ax_0 + (2a+1)y_0$$

which is written in the form of matrix as

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

where $M = \begin{pmatrix} 2a+1 & 2a+2 \\ 2a & 2a+1 \end{pmatrix}$

replacing the above process, the general solution (x_n, y_n) to (1) is given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

The eigen values of Mare $\alpha = (2a+1) + 2\sqrt{a^2+a}$ and $\beta = (2a+1) - 2\sqrt{a^2+a}$, it is well known that

$$M^n = \frac{\alpha^n}{\alpha-\beta}(M-\beta I) + \frac{\beta^n}{\alpha-\beta}(M-\alpha I)$$

Using the above formula, we have

$$M^n = \begin{pmatrix} \frac{\alpha^n+\beta^n}{2} & \frac{(a+1)}{2\sqrt{a^2+a}}(\alpha^n-\beta^n) \\ \frac{a}{2\sqrt{a^2+a}}(\alpha^n-\beta^n) & \frac{\alpha^n+\beta^n}{2} \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} Y_n & (a+1)X_n \\ aX_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

where

$$Y_n = \frac{1}{2} f_n \qquad , f_n = \alpha^n + \beta^n$$

$$X_n = \frac{1}{2\sqrt{a^2+a}} g_n, g_n = \alpha^n - \beta^n$$

Remarkable observations

Let (α, β, γ) be the sides of the Pythagorean triangle

$$\alpha = 2pq, \beta = p^2 - q^2, \gamma = p^2 + q^2, p > q > 0$$

where p and q are the generators of the Pythagorean triangle.

Let A and P be its area and perimeter respectively.

Write p and q as $p = x_n + y_n$ and $q = y_n$, where (x_n, y_n) is the solution of (1).

Then the corresponding Pythagorean triangle is such that

- $\gamma(a-1) - 2a\alpha + (a+1)\beta = 2a.$
- $aP^2 + P[(a+1)(\beta-\gamma) - 2a(\alpha+1)] = 4aA.$

CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

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