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HOPSCOTCH-CRANK-NICHOLSON-LAX FREDRICK (HP-CN-LF) HYBRID METHOD FOR SOLVING TWO DIMENSIONAL SYSTEM OF BURGERS' EQUATION

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ABSTRACT

In this research the hybrid Hopscotch Crank-Nicholson- Lax Fredich's method is developed to solve Burgers' equations. Here a hybrid Hopscotch-Crank-Nicholson-Lax Fredrich finite difference scheme is proposed to solve 2-D Burgers equations. Hybrid Hopscotch-Crank-Nicholson-Lax Fredrich Scheme (HP-CN-LF) compared well with earlier developed schemes. It has proved to be stable, consistent and convergent.

KEYWORD: Hopscotch, Burgers Equation, Crank-Nicholson-Lax-Fredrich.

1. INTRODUCTION

Let us consider the 2-D Burgers' equation is of the form:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ \frac{\partial v}{\partial t} &= -u\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{aligned}\right\} \quad (1)$$

The paper proposes a numerical solution to the system of Burgers equation (1).

2. Background information

Burgers' equation is a parabolic equation that has always been used as a mathematical model for many physical phenomena. In particular, it is widely used as a simplified model for understanding turbulence flow, boundary layer behavior, shock wave formation, convection dominated diffusion phenomena, acoustic attenuation in fog and continuum traffic simulation. Moreover, Burgers' equation is one of the very few nonlinear partial differential equations that can be solved exactly using a transformation for arbitrary initial and boundary conditions. Thus, the numerical method has practical significance, and has drawn the attention of many researchers in the past.

Several attempts have been made to solve Burgers' equation with varied level of accuracy. For example implicit methods, ADI methods, finite element methods, moving finite element, mixed finite element technique, Chebyshev spectral collocation methods and collocation procedures using cubic B-splines. In addition, the Group Explicit method, the odd-even Hopscotch scheme and the alternating direction implicit scheme to solve Burgers' equation on the Vector Machine, and spectral method using the finitely reproducing property of a nonlinear operator in order to solve Burgers' equation with different boundary conditions.

Caldwell and Wanless (1981) attempted a piecewise polynomial approximation also referred to as Finite Element where the size of the elements were chosen to take into account the nature of the solution. The aim was to 'chase the peak' by altering the size of the elements at each stage using information from the previous step. The research restricted attention to the use of piecewise polynomials, being the simplest form. The results were discussed and proved to be very satisfactory.

Evans and Abdullah (1984) proposed the Group Explicit method to the numerical solution of a non-linear parabolic partial differential equation of second order. The method was tested out on Burger's equation for various initial and boundary conditions. It proved that the method is accurate and comparable to existing finite difference methods.

The numerical values of the dependent variables are obtained at the points of intersection of the parallel lines, called mesh points or nodal points. These values are obtained by discretizing the governing partial differential equations over the region of interest to derive approximately equivalent algebraic equations. Discretization consists of replacing each derivative of the partial differential equation at mesh point by a finite difference approximation in terms of the values of the dependent variable at the mesh.

Bressan and Quarteron (1986) defined and analyzed Chebyshev spectral collocation methods for approximating the solution of Burgers' equation. Discretization in time by an implicit/explicit single step method was also discussed. The method was shown to be stable under a very weak condition on the time step, for the (linear) diffusive part is dealt with implicitly. Besides, fast transform methods was also used to compute the explicit (non-linear) convective term.

Hrymak *et. al.* (1986) explained the moving finite element method as an adaptive gridding procedure for systems of partial differential equations whose solutions contain steep gradients. The research implemented the method in a very straight forward way. The performance of the method was illustrated with solutions of Burgers' equation.

Sereno *et. al.* (1991) developed the moving finite element method (MFEM) using polynomial approximations of arbitrary degree in each of the finite elements. The approximations are then obtained by the Lagrange interpolation polynomials, with the interior nodes optimized as in the orthogonal collocation method. The research proposed that the method can be used for any type of linear boundary conditions. A computer code was developed to illustrate the method with three examples: the 1-D Burgers' problem; equilibrium model for fixed-bed adsorption; and pseudo-homogeneous axial dispersion model.

Samir (1999) solved numerically the 2-D unsteady coupled Burgers' equations with moderate to severe gradients, using higher-order accurate finite difference schemes; namely the fourth-order accurate compact ADI scheme, and the fourth-order accurate Du Fort Frankel scheme. The question of numerical stability and convergence were presented. Comparisons were made between the schemes in terms of accuracy and computational efficiency for solving problems with severe internal and boundary gradients. The study showed that the fourth-order compact ADI scheme is stable and efficient.

Chen and Jiang (2004) used a new mixed finite element method, called the characteristics mixed method, for approximating the solution to Burgers equation. The method is based upon a space-time variational form of Burgers' equation. The hyperbolic part of the equation is approximated along the characteristics in time and the diffusion part is approximated by a mixed finite element method of lowest order. The scheme is locally conservative since fluid is transported along the approximate characteristics on the discrete level and the test function can be piecewise constant. The new method approximated the scalar unknown and the vector flux optimally and simultaneously. The research showed that the scheme has much smaller time-truncation errors than those of standard methods. Numerical example was presented to show that the new scheme is easily implemented; shocks and boundary layers are handled with almost no oscillations.

Amir (2007) *et. al.* developed multi-symplectic box implicit methods for solving Burgers' equation. The research showed that the multi-symplectic box scheme is a very effective box scheme in diminishing artificial wiggles which appear in approximation solution. Two types of box schemes and implementation on the Burgers equation was done to get better results with no artificial wiggles.

Idris and Ali (2007) Iillustrated how the numerical solution of the Burgers' equation is obtained using the methods of cubic B-spline collocation and quadratic B-spline Galerkin over the geometrically graded mesh. The design involved partitioning of spatial domain into geometrically graded mesh. The finite element methods were constructed within the Galerkin and collocation methods using an expansion of the quadratic and cubic B-splines as an approximate function, respectively, over the mesh. The paper proved that higher errors are observed at near boundaries for shock-like and travelling wave solutions of the Burgers' equation when bigger mesh are used, accuracy of the defined methods increase by using finer mesh at near this boundary.

Shusen et. *al*. (2010) introduced a high-order accurate compact finite difference method using the Hopf–Cole transformation for solving 1-D Burgers' equation numerically. The stability and convergence analyses for the proposed method were given, and this method was shown to be unconditionally stable. To demonstrate efficiency, numerical results that were obtained by the proposed scheme were compared with the exact solutions and the results obtained by some other methods. Their proposed method is second- and fourth-order accurate in time and space, respectively. They derived a high-order accurate compact finite difference method (FDM) to numerically solve the linearized equation. The present method gives an implicit scheme with tri-diagonal symmetric positive-definite system, which could be easily implemented. Stability and convergence analyses showed that the method was unconditionally stable and has an accuracy of second- and fourth-order in time and space, respectively. Numerical experiments showed that the accuracy of the method and the fourth-order iterative Finite Difference Method is almost the same. The numerical solutions obtained by the method are in good agreement with the exact solutions, and their method gives compatible numerical results with the ones obtained by some other available methods given in references.

Al-Saif *et. al.* (2012) proposed a new development of differential quadrature method. It is known alternating direction implicit formulation of the differential quadrature method (ADI-DQM) for computing the numerical solutions of the two dimension Burger equations. The results confirm that this method has a high accuracy, good convergence and less workload comparing with the other numerical methods.

Vineet *et. al* (2013a) proposed a fully implicit finite-difference method for the numerical solutions of one dimensional coupled nonlinear Burgers' equations on the uniform mesh points. The method forms a system of nonlinear difference equations which was solved at each iteration. Newton's iterative method has been implemented to solve this nonlinear assembled system of equations. The linear system was solved by Gauss elimination method with partial pivoting algorithm at each iteration of Newton's method. Three test examples were carried out to illustrate the accuracy of the method. Computed solutions obtained by proposed scheme were compared with analytical solutions and those already available in the literature by finding L_2 and L_∞ errors.

Vineet *et. al* (2013b) described a new implicit finite-difference method: an implicit logarithmic finite-difference method (I-LFDM), for the numerical solution of two dimensional time-dependent coupled viscous Burgers' equation on the uniform grid points. As the Burgers' equation is nonlinear, the proposed technique leads to a system of nonlinear systems, which was solved by Newton's iterative method at each time step. Computed solutions were compared with the analytical solutions and those already available in the literature and it was clearly shown that the results obtained using the method is precise and reliable for solving Burgers' equation.

Vineet *et. al* (2013c) an implicit exponential finite-difference scheme (Expo FDM) was proposed for solving two dimensional nonlinear coupled viscous Burgers' equations (VBEs) with appropriate initial and boundary conditions. The accuracy of the method was illustrated by taking two numerical examples. Results were compared with exact solution and those already available in the literature by finding the L_1, L_2, L_∞ and E_R errors. Excellent numerical results indicate that the proposed scheme is efficient, reliable and robust technique for the numerical solutions of Burgers' equation.

Bilge and Ahmet (2013) proposed a numerical method to approximate the solution of the one-dimensional Burgers' equation. Technique called explicit exponential finite difference method was used. Since the Burgers' equation is nonlinear, the equation was converted to the linear heat equation by the Hopf-Cole transformation. And then, the explicit exponential finite difference method was applied to obtain numerical solution. The results were compared with exact values clearly showed that results obtained using the method were precise and reliable.

Vineet *et al*. (2014) implemented an implicit logarithmic finite difference method (I-LFDM) for the numerical solution of one dimensional coupled nonlinear Burgers' equation. The numerical scheme provided a system of nonlinear difference equations which they linearized using Newton's method. The obtained linear system via Newton's method was solved by Gauss elimination with partial pivoting algorithm. To illustrate the accuracy and reliability of the scheme, they described three numerical examples. The obtained numerical solutions proved to compare well with the exact solutions and those already available.

In this research hybrid Hopscotch Crank-Nicholson- Lax Fredich's method for solving two-dimensional system of Burgers' equations is developed. It is an extension of the work done by Maritim *et. al.* (2018) where the two dimensional Burgers equation is solved using hybrid Hopscotch-Crank-Nicholson-Du-Fort and Frankel scheme. In their research a solution algorithm for two-dimensional Burgers' equation with mixed boundary condition is derived then hybrid Hopscotch-Crank-Nicholson-Lax Fredrich Scheme finite scheme is developed.

In our research Hybrid Hopscotch-Crank-Nicholson-Lax Fredrich Scheme (HP-CN-LF) compared well with earlier developed schemes. It has proved to be stable, consistent and convergent. Here Lax-Frdrich method, which can be described as the FTCS (forward in time, centered in space) scheme with an artificial viscosity term of ½, is blended with Hopscotch and Crank-Nicholson.

3. Approximation at the Boundaries

Rotich *et. al*. (2016) proposed a solution of Burgers system of equations (1) to be is given by the following equations:

$$u = \frac{-2y-2\pi(exp-\frac{2\pi^2 t}{R})\,((cos\pi x-\sin\pi x)\sin\pi y)}{R(100+xy+(exp-\frac{2\pi^2 t}{R})\,((cos\pi x-\sin\pi x)\sin\pi y)} \tag{2}$$

$$v = \frac{-2x-2\pi(exp-\frac{2\pi^2 t}{R})\,((cos\pi x-\sin\pi x)\sin\pi y)}{R(100+xy+(exp-\frac{2\pi^2 t}{R})\,((cos\pi x-\sin\pi x)\sin\pi y)} \tag{3}$$

Using $\Delta x = \Delta y = h$ and $\Delta t/\,h^4 R = \beta$ and $\Delta t/h^2 = \alpha$ we obtain Hopscotch-Crank-Nicholson scheme as shown below, according to Maritim *et. al.* (2018).

$$\begin{aligned}3\beta U_{i-2,j}^{n+1} &+ U_{i-1,j}^{n+1}\left(\alpha U_{i-1,j-1}^{n}-6\beta\right)+U_{i,j}^{n+1}\left(1-2\alpha U_{i-1,j-1}^{n}+4\beta\right)+U_{i+1,j}^{n+1}\left(\alpha U_{i-1,j-1}^{n}-\beta\right)\\ &+3\beta U_{i,j-2}^{n+2}+U_{i,j-1}^{n+2}\left(\alpha V_{i-1,j-1}^{n}-6\beta\right)-U_{i,j}^{n+2}\left(2\alpha V_{i-1,j-1}^{n}-4\beta\right)\\ &+U_{i,j+1}^{n+2}\left(\alpha V_{i-1,j-1}^{n}-\beta\right)=U_{i,j}^{n}\end{aligned} \tag{4}$$

$$\begin{aligned}3\beta V_{i-2,j}^{n+1} &+ V_{i-1,j}^{n+1}\left(\alpha U_{i-1,j-1}^{n}-6\beta\right)+V_{i,j}^{n+1}\left(1-2\alpha U_{i-1,j-1}^{n}+4\beta\right)+V_{i+1,j}^{n+1}\left(\alpha U_{i-1,j-1}^{n}-\beta\right)\\ &+3\beta V_{i,j-2}^{n+2}+V_{i,j-1}^{n+2}\left(\alpha V_{i-1,j-1}^{n}-6\beta\right)-V_{i,j}^{n+2}\left(2\alpha V_{i-1,j-1}^{n}-4\beta\right)\\ &+V_{i,j+1}^{n+2}\left(\alpha V_{i-1,j-1}^{n}-\beta\right)=V_{i,j}^{n}\end{aligned} \tag{5}$$

4. Hybrid Schemes

4.1 Hopscotch-Crank-Nicholson-Lax-Fredrich's (HP-CN-LF) Hybrid Scheme

The Hopscotch Crank-Nicholson-Lax Fredich's Hybrid Scheme is obtained by replacing $U_{i,j}^{n}$ and $V_{i,j}^{n}$ by $\frac{1}{2}\left(U_{i+1,j+1}^{n}+U_{i-1,j+1}^{n}\right)+\frac{1}{2}\left(U_{i+1,j-1}^{n}+U_{i-1,j-1}^{n}\right)$ and $\frac{1}{2}\left(V_{i+1,j+1}^{n}+V_{i-1,j+1}^{n}\right)+\frac{1}{2}(V_{i+1,j-1}^{n}+V_{i-1,j-1}^{n})$ in the equation (4) and (5) respectively to get HP-CN-LF:

$$\begin{aligned}3\beta U_{i-2,j}^{n+1} &+ U_{i-1,j}^{n+1}\left(\alpha U_{i-1,j-1}^{n}-6\beta\right)+U_{i,j}^{n+1}\left(1-2\alpha U_{i-1,j-1}^{n}+4\beta\right)+U_{i+1,j}^{n+1}\left(\alpha U_{i-1,j-1}^{n}-\beta\right)\\ &+3\beta U_{i,j-2}^{n+2}+U_{i,j-1}^{n+2}\left(\alpha V_{i-1,j-1}^{n}-6\beta\right)-U_{i,j}^{n+2}\left(2\alpha V_{i-1,j-1}^{n}-4\beta\right)\\ &+U_{i,j+1}^{n+2}\left(\alpha V_{i-1,j-1}^{n}-\beta\right)=\frac{1}{2}\left(U_{i+1,j+1}^{n}+U_{i-1,j+1}^{n}\right)+\frac{1}{2}\left(U_{i+1,j-1}^{n}+U_{i-1,j-1}^{n}\right)\end{aligned} \tag{6}$$

$$3\beta V_{i-2,j}^{n+1} + V_{i-1,j}^{n+1}\left(\alpha U_{i-1,j-1}^{n} - 6\beta\right) + V_{i,j}^{n+1}\left(1 - 2\alpha U_{i-1,j-1}^{n} + 4\beta\right) + V_{i+1,j}^{n+1}\left(\alpha U_{i-1,j-1}^{n} - \beta\right) + 3\beta V_{i,j-2}^{n+2} + V_{i,j-1}^{n+2}\left(\alpha V_{i-1,j-1}^{n} - 6\beta\right) - V_{i,j}^{n+2}\left(2\alpha V_{i-1,j-1}^{n} - 4\beta\right) + V_{i,j+1}^{n+2}\left(\alpha V_{i-1,j-1}^{n} - \beta\right) = \frac{1}{2}\left(V_{i+1,j+1}^{n} + V_{i-1,j+1}^{n}\right) + \frac{1}{2}\left(V_{i+1,j-1}^{n} + V_{i-1,j-1}^{n}\right) \tag{7}$$

5. NUMERICAL RESULTS

Taking: $\Delta x = \Delta y = h = 0.1$, $\mathrm{R} = 4000$ and $\Delta t = k = 0.001$ to obtain the results.

Here we take: $0 \leq x, y \leq 1$ and the initial and boundary conditions for the velocity components u and v are taken from the proposed Kweyu *et. al*. (2012) scheme.

5.1 Absolute Errors in Solutions of u and v

Figures 1 and 2 show the absolute error in solutions of u and v respectively plotted against the x and y values for three hybrid schemes with $t = 1$.

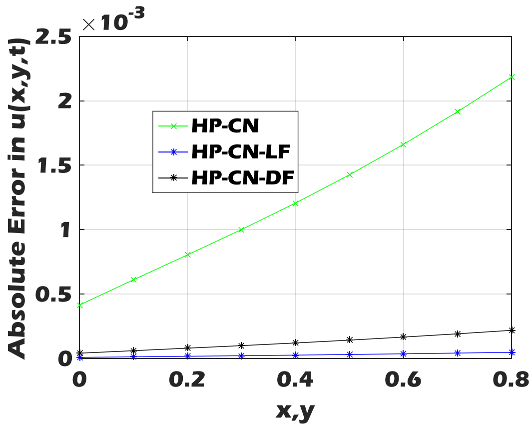

Figure 1: Absolute error in Solution of u for HP-CN, HP-CN-LF & HP-CN-DF

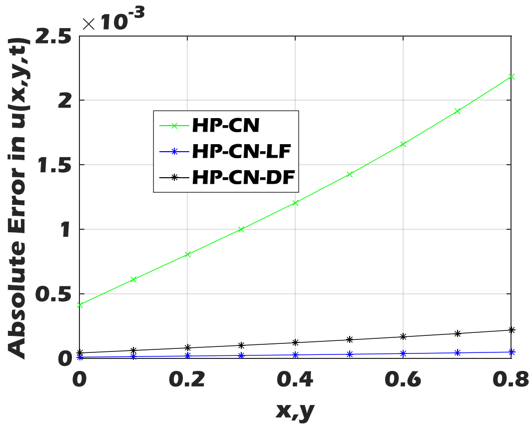

Figure 2: Absolute error in Solution of v for HP-CN, HP-CN-LF & HP-CN-DF.

The errors are the variation of the HP-CN, HP-CN-DF and HP-CN-LF when compared with values generated by the solution of the scheme proposed by Kweyu *et. al.* (2012). The figure clearly shows that the hybrid HP-CN-LF has the least error than HP-CN and HP-CN-DF. This shows that HP-CN-LF is the most accurate.

5.2 Solutions of u and $v$for HP-CN-LF

Figure 3 and 4 below shows a three dimensional plot of solutions of u and v when $t = 1$ for HP-CN-LF plotted against x and y variables.

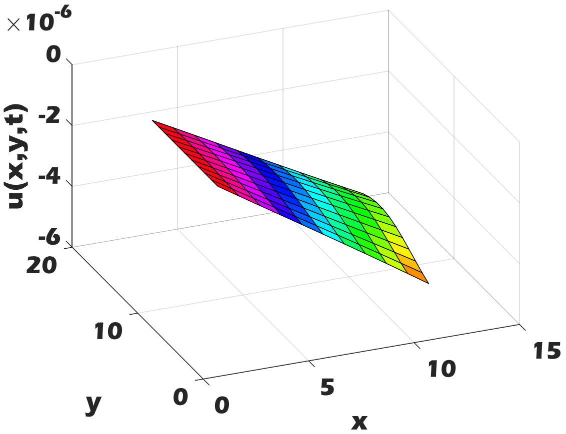

Figure 3: HP-CN-LF Solution of u at t=1.

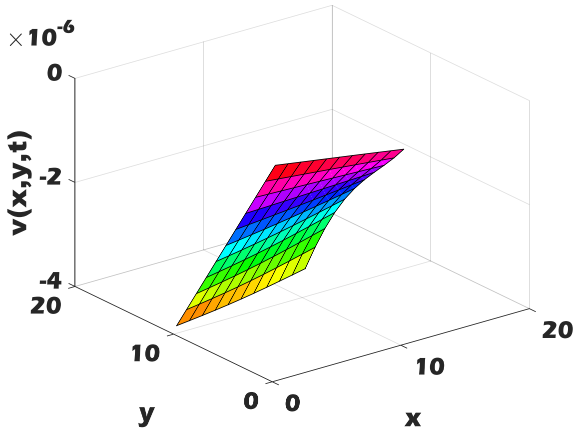

Figure 4: HP-CN-LF Solution of v at t=1

The figures shows that the results for the HP-CN-LF schemes developed are consistent.

Table 1: Solution of u for the 2-D Burgers equation at *t* = 1 for the different schemes.

X	Y	Kweyu *et. al.* (2012) Proposed scheme solution(*exp-0.006)	HP-CN (*exp-0.006)	HP-CN-DF (*exp-0.006)	HP-CN-LF (*exp-0.006)
0.1	0.1	-0.45210681629561	-0.45212809683860	-0.45210729075647	-0.45210468063855
0.2	0.2	-0.90666175924746	-0.90707726837174	-0.90667102335393	-0.90662005931782
0.3	0.3	-1.36301183011581	-1.36362293690697	-1.36302545536009	-1.36295049960099
0.4	0.4	-1.81993778965255	-1.82074210344674	-1.81995572275460	-1.81985706836629
0.5	0.5	-2.27602700265147	-2.27702783812104	-2.27604931747869	-2.27592655811940
0.6	0.6	-2.72986139439426	-2.73106779914542	-2.72988829257710	-2.72974031908276
0.7	0.7	-3.18025559498377	-3.18168135576848	-3.18028738376313	-3.18011250601133
0.8	0.8	-3.62646733993825	-3.62812910745055	-3.62650439036687	-3.62630056721496
0.9	0.9	-4.06834558851241	-4.07026042802845	-4.06838828086325	-4.06815342045898
1	1	-4.50639373341364	-4.50857649857835	-4.50644239865479	-4.50617468038073

Table 2: Solution of v for the 2-D Burgers equation at *t* = 1 for the different schemes.

X	y	Kweyu *et. al.*(2012) Proposed scheme solution(*exp-0.006)	HP-CN (*exp-0.006)	HP-CN-LF (*exp-0.006)	HP-CN-DF (*exp-0.006)
0.1	0.1	-4.91659648501478	-4.91661159189801	-4.91659682145726	-4.91659497061011
0.2	0.2	-4.93021949923746	-4.93053010322735	-4.93022641672314	-4.93018836200184
0.3	0.3	-4.89143745710773	-4.89192178335778	-4.89144824368021	-4.89138890414022
0.4	0.4	-4.84860678586601	-4.84928319580011	-4.84862185048505	-4.84853897632459
0.5	0.5	-4.80113941328106	-4.80202839659612	-4.80115921225686	-4.80105029321574
0.6	0.6	-4.74893329120019	-4.75005517398268	-4.74895827713826	-4.74882082337100
0.7	0.7	-4.69237703046003	-4.69374975282673	-4.69240760276388	-4.69223941713768
0.8	0.8	-4.63229606978619	-4.63393334321813	-4.63233253363041	-4.63213193741768
0.9	0.9	-4.56984942066285	-4.57175950062964	-4.56989195968453	-4.56965794277907
1	1	-4.50639373341364	-4.50857892118299	-4.50644239865479	-4.50617468038073

6. CONCLUSION

We have developed hybrid Hopscotch-Crank-Nicholson-Lax-Fredrich scheme (HP-CN-LF) to solve two-dimensional Burgers' equation. The errors are within acceptable range of less than 0.003%. The solution to the scheme is not changing suddenly with time and space. Thus our scheme is stable and convergent which implies consistency.

LIST OF ABBREVIATIONS

HP	:	Hopscotch
HP-CN	:	Hopscotch Crank-Nicholson
HP-CN-DF	:	Hopscotch Crank-Nicholson- Du-Fort and Frankel
HP-CN-LF	:	Hopscotch Crank-Nicholson-Lax-Friedrich's
R	:	Reynolds number
u	:	Fluid velocity in the x-direction
v	:	Fluid velocity in the y-direction.

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