



GEOMETRIC CONDITIONS FOR HOMOTHETIC-TO-PROJECTIVE CONVERSIONS IN FIFTH-ORDER RECURRENT BY LIE-DERIVATIVE IN FINSLER SPACES

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ABSTRACT

In this paper, we introduce five conditions for converting certain types of homothetic motions into projective motions in the generalized fifth recurrent Finsler space by Lie-derivative. Further prove that the Lie-derivative of the Cartan's fourth curvature tensor K_{jkh}^i and $h(v)$ -torsion tensor H_{kh}^i behave as fifth recurrent within this space that allows the conversion.

KEYWORDS: Projective motion, Homothetic motion, $GBK - SRF_n$, Berwald connection, Lie-derivative.

1. INTRODUCTION

A homothetic motion is credited to the German mathematician Gottlob Friedrich Helmholtz, who first studied the homothetic motion in (1857). The word “Hemothetice” is derived from the Greek word “Hemothesis” meaning “relative position”, it is used to describe geometric transformations that preserve size and direction. Hiramatu^[10] established a homothetic transformation in a Finsler space. Singh^[14] proved that every homothetic transformation in a

Finsler space is a W –curvature collineation. Opondo^[12] discussed the affine and homothetic motions in bi-recurrent Finsler space.

The projective changes of Finsler metrics and projectively flat Finsler space introduced by Matsumoto.^[11] Several theorems in Projective motion proved by Verma.^[15] Al-Qashbari et al.^[3-6] studied projective motion in generalized fifth recurrent Finsler space by Lie-derivative and established various identities on Lie-derivative of tensors within this space. Pandey^[13] studied Lie-derivative in recurrent Finsler space.

Several relations on curvature tensor of different order in Finsler space, and important results of generalized Finsler space in sense of Berwald established by Abdallah and et al.^[1,2,8,9] The necessary and sufficient condition for some tensors in generalized recurrent Finsler space introduced by.^[7] The aim of this paper is to study conversion a body's homothetic motion (around its own axis) into projective motion along its curved path (around gravitational source) in generalized fifth recurrent Finsler space by Lie-derivative. This allows for precisely determine the body's actual position in the projective path.

2. Preliminaries

A Lie - derivative evaluate the rate of change of a vector field or a tensor field along the smooth vector field $v^i(x)$. The Lie-derivative of a general mixed tensor field T_{jkh}^i is given by following formula

$$(2.1) \quad L_v T_{jkh}^i = v^m \mathcal{B}_m T_{jkh}^i - T_{jkh}^m \mathcal{B}_m v^i + T_{mkh}^i \mathcal{B}_j v^m + T_{jmh}^i \mathcal{B}_k v^m \\ + T_{jkm}^i \mathcal{B}_h v^m + \dot{\partial}_m T_{jkh}^i \mathcal{B}_r v^m y^r,$$

Where $v^m \neq 0$ is a contravariant vector field independent of directional argument and dependent on positional coordinates (x^i) only and $\mathcal{B}_j v^m = 0$.

The Lie-derivative of the metric tensor g_{ij} is vanishing, i.e.

$$(2.2) \quad L_v g_{ij} = 0.$$

The projective motion it is the motion of an object in curved space that is not effected by any forces other than gravity and moves along curved paths .the sufficient condition for motion to become projective is^[15]

$$(2.3) \quad L_v G_{kh}^i = \delta_k^i P_h + \delta_h^i P_k + y^i P_{kh}.$$

The homothetic motion it is the motion of an object in curved space where the object's angular velocity is constant at any point along its curved path. the sufficient condition for motion to become homothetic is^[12]:

$$(2.4) L_v g_{ij} = 2c g_{ij} \text{ (where } c \text{ is a constant).}$$

The Cartan's fourth curvature tensor K_{jkh}^i , Cartan's third curvature tensor R_{jkh}^i and $h(v)$ -torsion tensor H_{kh}^i satisfy the following relation

$$(2.5) K_{jkh}^i y^j = R_{jkh}^i y^j = H_{kh}^i.$$

The Ricci tensor P_{jk} and non-zero vector y^k satisfies the following relation

$$(2.6) P_{jk} y^k = 0.$$

The metric tensor g_{ij} and Kronecker delta δ_h^i are satisfying the relation

$$(2.7) g_{ij} g^{ik} = \delta_j^k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}.$$

In generalized fifth recurrent Finsler space for K_{jkh}^i in sense of Berwald ($GBK - 5RF_n$), some projective motions are given by^[3]

$$(2.8) L_v(\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jkh}^i) = K_{jkh}^i v^m \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m \mathcal{B}_m K_{jkh}^i + y^i P_{kh}.$$

$$(2.9) L_v(\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_{kh}^i) = H_{kh}^i v^m \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m \mathcal{B}_m H_{kh}^i + y^j y^i P_{kh}.$$

$$(2.10) P_{kh} = F^2 v^m \mathcal{B}_m P_{kh}.$$

$$(2.11) \mathcal{B}_m b_{sqlnm} = -\frac{y^j y^k P_{kh}}{y_h^j v^m}.$$

$$(2.12) L_v(\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m R_{jkh}^i) = L_v(\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jkh}^i),$$

if

$$(2.13) L_v(a_{sqlnm} R_{jkh}^i) = L_v(a_{sqlnm} K_{jkh}^i).$$

The Lie-derivative of the vector field b_{sqlnm} of fifth order vanishes identically, i.e.

$$(2.14) L_v b_{sqlnm} = 0.$$

The Berwald connection parameter G_{kh}^i vanish in $GBK - 5RF_n$, i.e.

$$(2.15) G_{kh}^i = 0.$$

The Lie-derivative and by Berwald's covariant derivative of fifth order for Cartan's fourth curvature tensor K_{jkh}^i are commutative, if the Lie-derivative of non-zero vector field of fifth

order a_{sqlnm} vanishes simultaneously with the vanishing of scalar function $\alpha(x)$ by Berwald's covariant derivative of fifth order, i.e.

$$(2.16) \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v K_{jkh}^i) = L_v (\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jkh}^i),$$

if

$$(2.17) L_v a_{sqlnm} = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m \alpha(x) = 0.$$

The Cartan's fourth curvature tensor K_{jkh}^i and the h(v)-torsion tensor H_{kh}^i within $GBK - 5RF_n$, are satisfying the relations

$$(2.18) \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jkh}^i = a_{sqlnm} K_{jkh}^i + b_{sqlnm} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) \\ - c_{sqlnm} (\delta_h^i C_{jkn} - \delta_k^i C_{jhn}) - d_{sqlnm} (\delta_h^i C_{jkl} - \delta_k^i C_{jhl}) \\ - e_{sqlnm} (\delta_h^i C_{jkq} - \delta_k^i C_{jhq}) - 2b_{qlnm} y^r \mathcal{B}_r (\delta_h^i C_{jks} - \delta_k^i C_{jhs})$$

and

$$(2.19) \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_{kh}^i = a_{sqlnm} H_{kh}^i + b_{sqlnm} (\delta_h^i y_k - \delta_k^i y_h).$$

The Lie-derivative and Berwald's covariant derivative of the vector y^j are vanishing identically, i.e.

$$(2.20) \text{ a) } L_v y^j = 0 \text{ and b) } \mathcal{B}_m y^j = 0.$$

3. Main Results

In this section, we established five conditions for conversion five certain Homothetic motions to projective motions in generalized fifth recurrent Finsler space by Lie-derivative.

Condition (1)

Theorem 3.1: In $GBK-5RF_n$, the condition (3.2) converts the homothetic motion (3.3) into the projective motion (2.8).

Proof: Using (2.7) and (2.15) in (2.3), then since $y^i \neq 0$, we get

$$(3.1) P_{kh} = 0.$$

Transvecting above equation by y^i and using (2.2) in the right side of result equation, we get

$$(3.2) y^i P_{kh} = L_v g_{kh}.$$

Using (2.4) in above equation, we get

$$(3.3) y^i P_{kh} = 2c g_{kh}.$$

Above equation represents a homothetic motion. Now using (2.4) and (2.2) in right side of above equation, then multiplying the result equation by (-1), we get

$$-y^i P_{kh} = 0.$$

Which can be written as

$$-y^i P_{kh} = L_v(a_{sqlnm} K_{jkh}^i) - L_v(a_{sqlnm} K_{jkh}^i).$$

Or

$$L_v(a_{sqlnm} K_{jkh}^i) = L_v(a_{sqlnm} K_{jkh}^i) + y^i P_{kh}.$$

Using (2.7) in (2.18), then using the result equation in the left side of above equation, we get

$$(3.4) \quad L_v(\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jkh}^i) = K_{jkh}^i L_v(a_{sqlnm}) + a_{sqlnm} L_v K_{jkh}^i + y^i P_{kh}.$$

Applying formula (2.1) to the tensors a_{sqlnm} and K_{jkh}^i , then using the result equations in above equation, we obtained the projective motion (2.8).

Condition (2)

Theorem 3.2: In $GBK-5RF_n$, the condition (3.5) converts the homothetic motion (3.6) into the projective motion (2.9).

Proof: Transvecting (3.1) by $y^j y^i$, then using (2.2) in the right side of the result equation, we get

$$(3.5) \quad y^j y^i P_{kh} = L_v g_{kh}.$$

Using (2.4) in above equation, we get

$$(3.6) \quad y^j y^i P_{kh} = 2c g_{kh}.$$

Above equation represents a homothetic motion. Now using (2.4) and (2.2) in right side of above equation, then multiplying the result equation by (-1), we get

$$-y^j y^i P_{kh} = 0.$$

Which can be written as

$$-y^j y^i P_{kh} = L_v(a_{sqlnm} H_{kh}^i) - L_v(a_{sqlnm} H_{kh}^i).$$

Or

$$L_v(a_{sqlnm} H_{kh}^i) = L_v(a_{sqlnm} H_{kh}^i) + y^j y^i P_{kh}.$$

Using (2.7) in (2.19), then using the result equation in the left side of above equation, we get

$$(3.7) \quad L_v(\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_{kh}^i) = H_{kh}^i L_v(a_{sqlnm}) + a_{sqlnm} L_v H_{kh}^i + y^j y^i P_{kh}.$$

Applying formula (2.1) to the tensors a_{sqlnm} and H_{kh}^i , then using the result equations in above equation, we obtained the projective motion (2.9).

Now, we have two corollaries related to the previous theorems. Using (2.16) and (3.1) in (3.4), we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v K_{jkh}^i) = a_{sqlnm} (L_v K_{jkh}^i).$$

Above equation means the Lie-derivative of the Cartan's fourth curvature tensor K_{jkh}^i behaves as fifth recurrent. Thus, we conclude

Corollary 3.1: *In the main space that allows the conversion of homothetic motion (3.3) into the projective motion (2.8), the Lie-derivative of the Cartan's fourth curvature tensor K_{jkh}^i behaves as fifth recurrent [provided (2.17) holds].*

Using (2.16) and (3.1) in (3.7), we get

$$(3.8) \quad L_v(\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_{kh}^i) = a_{sqlnm} (L_v H_{kh}^i).$$

Transvecting (2.16) by y^j , using [(2.20)a,b] and (2.5), we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v H_{kh}^i) = L_v (\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_{kh}^i).$$

Using above equation in (3.8), we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v H_{kh}^i) = a_{sqlnm} (L_v H_{kh}^i).$$

Above equation means the Lie-derivative of the $h(v)$ -torsion tensor H_{kh}^i behaves as fifth recurrent. Thus, we conclude

Corollary 3.2: *In the main space that allows the conversion of homothetic motion (3.6) into the projective motion (2.9), the Lie-derivative of the $h(v)$ -torsion tensor H_{kh}^i behaves as fifth recurrent [provided (2.17) holds].*

Condition (3)

Theorem 3.3: *In $GBK-5RF_n$, the condition (3.9) converts the homothetic motion (3.10) into the projective motion (2.10).*

Proof: Taking the Lie-derivative of both sides of (3.1), then using (2.2) in the right side of the result equation, we get

$$L_v P_{kh} = L_v g_{kh}.$$

Transvecting above equation by the quadrature fundamental function of Finsler space F^2 , we get

$$F^2 L_v P_{kh} = F^2 L_v g_{kh}.$$

Applying formula (2.1) to the P -Ricci tensor P_{kh} , then using the result equations in above equation, we

$$(3.9) \quad F^2 v^m B_m P_{kh} = F^2 L_v g_{kh}.$$

Using (2.4) in above equation, we get

$$(3.10) \quad F^2 v^m B_m P_{kh} = 2c(F^2 g_{kh}).$$

Above equation represents a homothetic motion. Now using (2.4), (2.2) and (3.1) in right side of above equation, we get

$$F^2 v^m B_m P_{kh} = P_{kh} - 2c g_{kh}.$$

Which can be written as

$$P_{kh} - F^2 v^m B_m P_{kh} = 2c g_{kh}.$$

Using (2.4) and (2.2) in above equation, we obtained the projective motion (2.10).

Condition (4)

Theorem 3.4: In $GBK-5RF_n$, the condition (3.11) converts the homothetic motion (3.12) into the projective motion (2.11).

Proof: Transvecting (2.14) by y_h , then using (2.2) in right side of above equation, we get

$$(3.11) \quad y_h L_v b_{sqlnm} = L_v g_{kh}.$$

Using (2.4) in above equation, we get

$$(3.12) \quad y_h L_v b_{sqlnm} = 2c g_{kh}.$$

Above equation represents a homothetic motion. Now using (2.4) and (2.2) above equation, we get

$$y_h L_v b_{sqlnm} = 0.$$

Transvecting (3.1) by $(-y^j y^k)$, then using the result equation in above equation, we get

$$y_h L_v b_{sqlnm} = -y^j y^k P_{kh}.$$

Applying formula (2.1) to the vector field of fifth order b_{sqlnm} , then using the result equations in above equation, we obtained the projective motion (2.11).

Condition (5)

Theorem 3.5: In $GBK-5RF_n$, the condition (3.14) converts the homothetic motion (3.15) into the projective motion (2.12).

Proof: In view of (2.5), since $y^j \neq 0$, we get

$$(3.13) \quad R_{jkh}^i = K_{jkh}^i.$$

Using (2.2) in above equation, we get

$$(3.14) \quad R_{jkh}^i - K_{jkh}^i = L_v g_{kh}.$$

Using (2.4) in above equation, we get

$$(3.15) \quad R_{jkh}^i - K_{jkh}^i = 2c g_{kh}.$$

Above equation represents a homothetic motion. Now taking the Berwald covariant derivative of fifth order for above equation with respect to x^m, x^n, x^l, x^q and x^s respectively, and using (2.4) and (2.2), we get

$$(3.16) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m R_{jkh}^i = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jkh}^i.$$

Using (2.7) in (2.18), we get

$$(3.17) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jkh}^i = a_{sqlnm} K_{jkh}^i.$$

Transvecting (2.5) by a_{sqlnm} , we get

$$a_{sqlnm} K_{jkh}^i = a_{sqlnm} R_{jkh}^i.$$

Using (3.17) and above equation in (3.16), we get

$$(3.18) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m R_{jkh}^i = a_{sqlnm} R_{jkh}^i.$$

Using (3.17) and (3.18) in condition (2.13), we obtained the projective motion (2.12).

4. CONCLUSIONS

Five conditions convert certain homothetic motions into projective motions in $GBK-5RF_n$ by Lie-derivative have been discussed. Also, we obtained certain types of tensors with Lie-derivative that behave as fifth recurrent in the main space.

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