# STRONG RESULT \& ASYMPTOTIC ESTIMATES OF REAL ZEROS OF RANDOM POLYNOMIALS 

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#### Abstract

This paper provides asymptotic estimates strong result for real zeros of random algebraic polynomial for the expected number of real zeros of a random algebraic polynomial of the form The strong result for the lower bound was obtained in the general case by Their lower bound was $\frac{\mu \log n}{\log \left\{\frac{k_{n}}{t_{n}} \log n\right\}}$ Which is obtained by taking $\varepsilon_{n}=\mu / \log \left\{\frac{k_{n}}{t_{n}} \log n\right\}$ in our present result.


 This result is better than that of Dunnage since our constant is $(1 / \sqrt{ } 2)$. Times his constant and our error term is smaller. The proof is based on the convergence of an integral of which an asymptotic estimation is obtained. 1991 Mathematics subject classification (amer. Math. Soc.): 60 B 99.KEYWORDS AND PHRASES: Independent, identically distributed random variables, random algebraic polynomial, random algebraic equation, real roots, domain of attraction of the normal law, slowly varying function.

## INTRODUCTION

We shall suppose that's $\xi_{v}(\omega)$ real-valued random variables defined on the probability space ( $\Omega, \mathrm{m}, \mathrm{P}$ ). The random events to be considered in the proof correspond to P -measurable subsets of this space. The probability that an event E occurs will be denoted by $P(E)$.

Let $N_{n}$ be the number of real roots of $f(x, \omega) \sum_{v=0}^{n} \xi_{v}(\omega) x^{v}$. In Mishra, Nayak and Pattnayak, $\mathrm{S} .{ }^{[5]}$ we have shown that for $n>n_{0}, N_{n}$ is at least $\varepsilon_{n} \log n$ outside an exceptional set of measure at most $\mu /\left(\varepsilon_{n_{0}} \log n_{0}\right)$ where $\left\{\varepsilon_{n}\right\}$ is any sequence tending to zero such that $\varepsilon_{n}^{2} \log n$ tends to infinity as n tends to infinity. We have assumed that the $\xi_{v}$ 's have a common characteristics function $\exp (-C|t| \alpha)$ where $\alpha \geq 1$ and C is a positive constant.

In the present work we have proved the same result in the general case. We assume that the $\xi_{v}$ 's are any random variables with finite variance and third absolute moment. Our previous result holds in the case of a special characteristic function which has infinite variance $(1<\alpha<2)$.

The strong result for the lower bound was obtained in the general case by Samal and Mishra. ${ }^{[4]}$ Their lower bound was

$$
\frac{\mu \log n}{\log \left\{\frac{k_{n}}{t_{n}} \log n\right\}}
$$

Which is obtained by taking $\varepsilon_{n}=\mu / \log \left\{\frac{k_{n}}{t_{n}} \log n\right\}$ in our present result,

Where $k_{n}, t_{n}$ have the same meaning as in our present work.
We claim that our strong result for the lower bound in the general case is the best estimation done so far.

We shall use $[\mathrm{x}]$ to denote the greatest integer not exceeding x .
2. THEOREM 1. Let $f(x, \omega)=\sum_{v=0}^{n} \xi_{v}(\omega) x^{v}$ be a polynomial of degree n whose coefficients are independent random variables with expectation zero. Let $\sigma_{v}^{2}$ be the variance and $\tau_{v}^{3}$ be the third absolute moment of $\xi_{v}(\omega)$.Take $\left\{\varepsilon_{n}\right\}$ to be a sequence tending to zero such that $\varepsilon_{n}^{2} \log n$ tends to infinity as $n$ tends to infinity. Let $t_{n}=\min _{0 \leq v \leq n} \sigma_{v}, k_{n}=\max _{0 \leq v \leq n} \sigma_{v}$ and $p_{n}=\max _{0 \leq v \leq n} \tau_{v}$.Then there exists an integer $n_{0}$ and a set $A(\omega)$ of measure at most

$$
\mu / \varepsilon_{n_{0}} \log n_{0}
$$

such that, for $n>n_{0}$ and all $\omega$ not belonging to $A(\omega)$, the equations $f(x, \omega)=0$ have at least $\varepsilon_{n} \log n$ real roots, provided $\lim \frac{p_{n}}{k_{n}}$ and $\lim \frac{k_{n}}{t_{n}}$ are finite.

## Preliminary lemmas.

LEMMA 1. Suppose $X_{1}, X_{2}, \ldots X_{n}$ are independent random variables with expectation zero, and that $A_{v}^{2}$ is the variance and $B_{v}^{3}$ is the third absolute moment of $X_{v}$.

Let

$$
\mu_{n}^{2}=\sum_{v=1}^{n} A_{v}^{2}, \quad \lambda_{v}=\left\{\begin{array}{l}
\frac{B_{v}^{3}}{A_{v}^{2}} \text { if } A_{v} \neq 0 \\
0 \\
\text { if } A_{v}=\Lambda_{n}=\max _{1 \leq v \leq n}\left(\lambda_{n}\right)
\end{array}\right.
$$

Also let $F_{n}(t)$ be the distribution function of $\frac{1}{\mu_{n}} \sum_{v=1}^{n} X_{v}$ and

$$
\phi(t)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{t} \exp \left(-\frac{1}{2} u^{2}\right) d u
$$

Then

$$
\sup _{t}\left|F_{n}(t)-\phi(t)\right| \leq 2\left(\Lambda_{n} / \mu_{n}\right)
$$

This result is due to Esseen ${ }^{[2]}$ and Berry ${ }^{[1]}$

LEEMA 2 Let $\eta_{1}, \eta_{2}, \eta_{3}, \ldots$ be a sequence of independent random variables identically distributed with $V\left(\eta_{i}\right)<1$ for all i. Then, for each $\varepsilon>0$

$$
p\left\{\sup _{k \geq k_{0}}\left|\frac{1}{k} \sum_{i=1}^{k}\left\{\eta_{i}-E\left(\eta_{i}\right)\right\}\right| \geq \varepsilon\right\} \leq \frac{D}{\varepsilon^{2} k_{0}^{\prime}}
$$

Where D is a positive constant. This form of the strong law of large number is a consequence of the Hajek-Renyi inequality (see [3]).

Proof of the theorem. Take $\beta_{n}=\frac{t_{n}}{k_{n}} \exp \left\{\frac{C_{1}}{\varepsilon_{n}^{2} \log n}\right\}$

Where $\mathrm{C}_{1}$ is a constant to be chosen later.
Let $A$ and $B$ be constants such that $0<B<1$ and $A>1$. Let

$$
M_{n}=\left[2 \beta_{n}^{2}\left(\frac{k_{n}}{t_{n}}\right)^{2} \frac{A e}{B}\right]+1 .
$$

(2.1)

So $\quad \mu\left(\frac{k_{n}}{t_{n}}\right)^{2} \beta_{n}^{2} \leq M_{n} \leq \mu^{\prime}\left(\frac{k_{n}}{t_{n}}\right)^{2} \beta_{n}^{2}$.
We define

$$
\phi(x)=x^{[\log x]+x}
$$

Let k be the integer determined by

$$
\begin{equation*}
\phi(8 k+7) M_{n}^{8 k+7} \leq n<\phi(8 k+11) M_{n}^{8 k+11} . \tag{2.2}
\end{equation*}
$$

Obviously

$$
\mu_{1} \sqrt{\frac{\log n}{\log \left(\frac{k_{n}}{t_{n}} \beta\right)}} \leq k \leq \frac{\mu_{2} \log n}{\log \left(\frac{k_{n}}{t_{n}} \beta_{n}\right)}
$$

(2.3)
which implies

We consider

$$
\frac{\mu_{1}}{\sqrt{C_{1}}} \varepsilon_{n} \log n \leq k \leq \frac{\mu_{2}}{C_{1}}\left(\varepsilon_{n} \log n\right)^{2} .
$$

$$
f(x, \omega)=U_{m}(\omega)+R_{m}(\omega)
$$

at the points

$$
\begin{equation*}
x_{m}=\left\{1-\frac{1}{\phi(4 m+1) M_{n}^{4 m}}\right\}^{1 / 2} \tag{2.4}
\end{equation*}
$$

for $m=\lfloor k / 2\rfloor+1,\lfloor k / 2\rfloor+2, . . k$, where

$$
U_{m}(\omega)=\sum_{1} \xi_{v}(\omega) x_{m}^{v}, R_{m}(\omega)=\left(\sum_{2}+\sum_{3}\right) \xi_{v}(\omega) x_{m}^{v}
$$

the index $v$ ranging from $\phi(4 m-1) M_{n}^{4 m-1}+1$ to $\phi(4 m+3) M_{n}^{4 m+3}$ in $\sum_{1}$, from 0 to $\phi(4 m-1) M_{n}^{4 m-1}$ in $\sum_{2}$, and from $\phi(4 m+3) M_{n}^{4 m+3}+1$ to $n$ in $\sum_{3}$.

Let

$$
V_{m}=\frac{1}{2}\left(\sum_{1} \sigma_{v}^{2} x_{m}^{2 v}\right)^{1 / 2}
$$

We define the events $E_{m}$ as the sets of $\omega$ for which $U_{2 m}(\omega)>V_{2 m}$ and $U_{2 m+1}(\omega)<-V_{2 m+1}$ and the events $F_{m}$ as the sets of $\omega$ for which $U_{2 m}(\omega)<-V_{2 m}$ and $U_{2 m+1}(\omega)>V_{2 m+1}$. Obviously the sets of $\xi_{v}{ }^{\prime} s$ in $U_{2 m}(\omega)$ and the sets of $\xi_{v}{ }^{\prime} s$ in $U_{2 m+1}(\omega)$ are disjoint. Thus $U_{2 m}(\omega)$ and $U_{2 m+1}(\omega)$ are independent random variables.

Let $S_{m}^{+}, S_{m}^{-}$be the sets of $\omega$ in which respectively $U_{m}(\omega)>V_{m}, U_{m}(\omega)<-V_{m}$.
Hence $E_{m} \cup F_{m}=\left(S_{2 m}^{+} \cap S_{2 m+1}^{-}\right) \cup\left(S_{2 m}^{-} \cap S_{2 m+1}^{+}\right)$.

Since the two sets within the braces on the right hand side are disjoint and since $U_{2 m}(\omega)$ and $U_{2 m+1}(\omega)$ are independent random variables,
$P\left(E_{m} \cup F_{m}\right)=P\left(S_{2 m}^{+}\right) P\left(S_{2 m+1}^{-}\right)+P\left(S_{2 m}^{-}\right) P\left(S_{2 m+1}^{+}\right)$.
If $\sigma^{2}$ is the variance of $U_{2 m}(\omega)$ then $\sigma^{2}=4 V_{2 m}^{2}$.

So $\sigma=2 V_{2 m}$. Let $F_{2 m}(t)$ be the distribution function of $\frac{U_{2 m}(\omega)}{\sigma}$.

Hence $P\left\{U_{2 m}(\omega)<-V_{2 m}\right\}=P\left\{U_{2 m}(\omega) / \sigma<-\frac{1}{2}\right\}=F_{2 m}\left(-\frac{1}{2}\right)$.

Here we shall apply Lemma 1.

In our case $B_{v}^{3}=\tau_{v}^{3} x_{2 m}^{3 v}, A_{v}^{2}=\sigma_{v}^{2} x_{2 m}^{2 v}$.

So

$$
\lambda_{v}=\left(\tau_{v}^{3} / \sigma_{v}^{2}\right) x_{2 m}^{v},
$$

$$
\Lambda_{n}=\max _{0 \leq v \leq n}\left(\tau_{v}^{3} / \sigma_{v}^{2}\right) x_{2 m}^{v} \leq \frac{p_{n}^{3}}{t_{n}^{2}}
$$

And

$$
\mu_{n}=\sigma=2 V_{2 m} .
$$

Therefore

$$
\sup _{t}\left|F_{2 m}(t)-\phi(t)\right| \leq \frac{p_{n}^{3}}{t_{n}^{2}} \frac{1}{V_{2 m}}
$$

Hence

$$
P\left(S_{2 m}^{-}\right)=F_{2 m}\left(-\frac{1}{2}\right) \geq \phi\left(-\frac{1}{2}\right)-\left|F_{2 m}\left(-\frac{1}{2}\right)-\phi\left(-\frac{1}{2}\right)\right|>\phi\left(-\frac{1}{2}\right)-\frac{p_{n}^{3}}{t_{n}^{2}} \frac{1}{V_{2 m}} .
$$

Similarly the other probabilities can be calculated.
Therefore

$$
P\left(E_{m} \cup F_{m}\right) \geq\left\{1-\phi\left(\frac{1}{2}\right)-\frac{p_{n}^{2}}{t_{n}^{2} V_{2 m}}\right\}\left\{\phi\left(-\frac{1}{2}\right)-\frac{p_{n}^{3}}{t_{n}^{2} V_{2 m+1}}\right\}+\left\{\phi\left(-\frac{1}{2}\right)-\frac{p_{n}^{3}}{t_{n}^{2} V_{2 m}}\right\}\left\{1-\phi\left(\frac{1}{2}\right)-\frac{p_{n}^{3}}{t_{n}^{2} V_{2 m+1}}\right\}
$$

It can be easily shown as in ${ }^{[5]}$ that

$$
\begin{equation*}
V_{m}^{2}>\frac{t_{n}^{2}}{4} \phi(4 m+1) M_{n}^{4 m}(B / A) e^{-1} \tag{2.5}
\end{equation*}
$$

When n is large.
So

$$
V_{m}^{2}>\frac{t_{n}^{2}}{4}(8 m+1) M_{n}^{8 m}(B / A) e^{-1}
$$

The least value of m is $[\mathrm{k} / 2]+1$. Hence $V_{2 m}>t_{n} A_{n}$
where $A_{n} \rightarrow \infty$ as $n \rightarrow \infty$, since $M_{n}>1$ and $8 m+1>\mu k>\mu^{\prime} \varepsilon_{n} \log n$.
Since $\lim _{n \rightarrow \infty}\left(p_{n} / t_{n}\right)$ is finite, it follows that

$$
\frac{p_{n}^{3}}{t_{n}^{2} V_{2 m}}<\frac{p_{n}^{3}}{t_{n}^{3}} \frac{1}{A_{n}}
$$

Tends to zero as n tends to infinity.
Therefore $P\left(E_{m} \cup F_{m}\right)$ is greater than a quantity which tends to $2 \phi\left(-\frac{1}{2}\right)\left\{1-\phi\left(\frac{1}{2}\right)\right\}$ as n tends to infinity.

Denote this last expression by $\delta$.

LEMMA 3 There is a set $\Omega_{m}$ of measure at most

$$
\frac{1}{m^{2} \beta_{n}^{2}}+\frac{16 A e}{B}\left(\frac{k_{n}}{t_{n}}\right)^{2} \exp \left\{-(4 m+1)^{2} M_{n}^{2}\right\}
$$

such that if $\omega \notin \Omega_{m}$ and $n>n_{0}$ then

$$
\left|R_{m}(\omega)\right|<V_{m}
$$

for $m=\lfloor k / 2\rfloor+1,\lfloor k / 2\rfloor+2, . . k$.

## Proof

$$
R_{m}(\omega)=\left(\sum_{2}+\sum_{3}\right) \xi_{v}(\omega) x_{m}^{v} .
$$

By Tchebycheff's inequality, we have

$$
P\left\{\left|\sum_{3} \xi_{v}(\omega) x_{m}^{v}\right| \geq \frac{1}{2} V_{m}\right\} \leq \frac{4 k_{n}^{2}}{V_{m}^{2}} \sum_{3} x_{m}^{2 v} .
$$

Proceeding as in Lemma 2.2 of [5], we now get that the above probability does not exceed

$$
\frac{16 A e}{B}\left(\frac{k_{n}}{t_{n}}\right)^{2} \exp \left\{-(4 m+1)^{2} M_{n}^{2}\right\} .
$$

Again, by using the same inequality

$$
P\left\{\left|\sum_{2} \xi_{v}(\omega) x_{m}^{v}\right|>m \beta_{n}\left(\sum_{2} \sigma_{2}^{v} x_{m}^{2 v}\right)^{1 / 2}\right\}<\frac{1}{m^{2} \beta_{n}^{2}} .
$$

Thus if $\omega \notin \Omega_{m}$ where

$$
P\left(\Omega_{m}\right)<\frac{1}{m^{2} \beta_{n}^{2}}+\frac{16 A e}{B}\left(\frac{k_{n}}{t_{n}}\right)^{2} \exp \left\{-(4 m+1)^{2} M_{n}^{2}\right\}
$$

we have

$$
\left|R_{m}(\omega)\right|<\frac{1}{2} V_{m}+m \beta_{n}\left(\sum_{2} \sigma_{v}^{2} x_{m}^{2 v}\right)^{1 / 2}
$$

Now , by using (2.1) and (2.5) and following the procedure of Lemma 2.3 of [5], we have

$$
m \beta_{n}\left(\sum_{2} \sigma_{v}^{2} x_{m}^{2 v}\right)^{1 / 2}<\frac{1}{2} V_{m}
$$

We have shown earlier that

$$
P\left(E_{m} \cup F_{m}\right)=\delta_{m}>\delta>0 .
$$

Let $\eta_{m}$ be a random variable such that it takes value 1 on $E_{m} \cup F_{m}$ and zero elsewhere. In other words

$$
\eta_{m}=\left\{\begin{array}{l}
1 \text { with probability } \delta_{m} \\
0 \text { with probability } 1-\delta_{m}
\end{array}\right.
$$

The $\eta_{m}$ 's are thus independent random variables with $E\left(\eta_{m}\right)=\delta_{m}$ and $V\left(\eta_{m}\right)=\delta_{m}-\delta_{m}^{2}<1$.

Let $\rho_{m}$ be defined as follows:

$$
\rho_{m}=\left\{\begin{array}{l}
0 \text { if }\left|R_{2 m}(\omega)\right|<V_{2 m} \text { and }\left|R_{2 m+1}(\omega)\right|<V_{2 m+1} \\
1 \text { otherwise. }
\end{array}\right.
$$

Let $\quad \theta_{m}=\eta_{m}-\eta_{m} \rho_{m}$.

The conclusion of section 2.4 of [5] gives that the number of roots in $\left(x_{2 m_{0}}, x_{2 k+1}\right)$ must exceed $\sum_{m=m_{0}}^{k} \theta_{m}$

Where $m_{0}=\left[\frac{1}{2} k\right]+1$.
2.4. Now we appeal to Lemma 2.

We have $\quad\left|\sum_{m=m_{0}}^{k}\left\{\theta_{m}-E\left(\eta_{m}\right)\right\}\right| \leq\left|\sum_{m=m_{0}}^{k}\left\{\eta_{m}-E\left(\eta_{m}\right)\right\}\right|+\sum_{m=m_{0}}^{k} \rho_{m}$.

Let $A(\omega)$ be the set of $\omega$ for which

$$
\sup _{k-m_{0}+1 \geq k_{0}} \frac{1}{k-m_{0}+1}\left|\sum_{m=m_{0}}^{k}\left\{\theta_{m}-E\left(\eta_{m}\right)\right\}\right|>\varepsilon,
$$

$B(\omega)$ be the set of $\omega$ for which

$$
\sup _{k-m_{0}+1 \geq k_{0}} \frac{1}{k-m_{0}+1}\left|\sum_{m=m_{0}}^{k}\left\{\eta_{m}-E\left(\eta_{m}\right)\right\}\right|>\frac{1}{2} \varepsilon
$$

and $C(\omega)$ be the set of $\omega$ for which

$$
\begin{aligned}
& \sup _{k-m_{0}+1 \geq k_{0}} \frac{1}{k-m_{0}+1} \sum_{m=m_{0}}^{k} \rho_{m}>\frac{1}{2} \varepsilon . \\
& E\left(\rho_{m}\right)=P\left\{\left(\left|R_{2 m}\right| \geq V_{2 m}\right) \cup\left(\left|R_{2 m+1}\right| \geq V_{2 m+1}\right)\right\} \leq P\left(\left|R_{2 m}\right| \geq V_{2 m}\right)+P\left(\left|R_{2 m+1}\right| \geq V_{2 m+1}\right) .
\end{aligned}
$$

By Lemma 3,

$$
P\left(\left|R_{2 m}\right| \geq V_{2 m}\right)<\frac{1}{4 m^{2} \beta_{n}^{2}}+\frac{16 A e}{B}\left(\frac{k_{n}}{t_{n}}\right)^{2} \exp \left\{-(8 m+1)^{2} M_{n}^{2}\right\}<\frac{1}{m^{2} \beta_{n}^{2}}+\frac{16 A e}{B}\left(\frac{k_{n}}{t_{n}}\right)^{2} \exp \left(-m^{2} M_{n}^{2}\right) .
$$

Similarly

$$
P\left(\left|R_{2 m+1}\right| \geq V_{2 m+1}\right)<\frac{1}{m^{2} \beta_{n}^{2}}+\frac{16 A e}{B}\left(\frac{k_{n}}{t_{n}}\right)^{2} \exp \left(-m^{2} M_{n}^{2}\right) .
$$

Hence by using (2.1), we have

$$
E\left(\rho_{m}\right)<\frac{\mu}{m^{2} \beta_{n}^{2}}+\mu^{\prime}\left(\frac{k_{n}}{t_{n}}\right) \exp \left(-m^{2} M_{n}^{2}\right)<\mu^{\prime \prime} /\left(m^{2} \beta_{n}^{2}\right)<\mu^{\prime \prime} / m^{2} .
$$

Therefore

$$
\frac{1}{k-m_{0}+1} \sum_{m+m_{0}}^{k} E\left(\rho_{m}\right)<\mu^{\prime \prime} / m_{0}^{2}
$$

and so

$$
P\{C(\omega)\}<\sum_{k-m_{0}+1 \geq k_{0}} P\left\{\frac{1}{k-m_{0}+1} \sum_{m=m_{0}}^{k} \rho_{m}>\frac{1}{2} \varepsilon\right\}<\frac{2 \mu^{\prime \prime}}{\varepsilon} \sum_{k-m_{0}+1 \geq k_{0}} \frac{1}{m_{0}^{2}} .
$$

Again by Lemma 2, we have

$$
P\{B(\omega)\}<\frac{4 D}{\varepsilon^{2} k_{0}} .
$$

Since

$$
\sup _{k-m_{0}+1 \geq k_{0}} \frac{1}{k-m_{0}+1}\left|\sum_{m=m_{0}}^{k}\left\{\theta_{m}-E\left(\eta_{m}\right)\right\}\right| \leq \sup _{k-m_{0}+1 \geq k_{0}} \frac{1}{k-m_{0}+1}\left|\sum_{m=m_{0}}^{k}\left\{\eta_{m}-E\left(\eta_{m}\right)\right\}\right|+\sup _{k-m_{0}+1 \geq k_{0}} \frac{1}{k-m_{0}+1} \sum_{m=m_{0}}^{k} \rho_{m}
$$

it follows that

$$
A(\omega) \subseteq B(\omega) \cup C(\omega)
$$

Hence calculation as in ${ }^{[5]}$ gives

$$
P\{A(\omega)\}<\frac{\mu^{\prime}}{k_{0}}<\frac{\mu^{\prime \prime}}{\varepsilon_{n_{0}} \log n_{0}} .
$$

Thus if

$$
\begin{aligned}
& \omega \notin A(\omega) \\
& \frac{1}{k-m_{0}+1} \sum_{m=m_{0}}^{k} \theta_{m}>\frac{1}{k-m_{0}+1} \sum_{m=m_{0}}^{k} E\left(\eta_{m}\right)-\varepsilon
\end{aligned}
$$

for all k such that $k-m_{0}+1 \geq k_{0}$.
So

$$
N_{n}>\frac{1}{2}(\delta-\varepsilon) k>\frac{1}{2}(\delta-\varepsilon) \frac{\mu_{1}}{\sqrt{c_{1}}} \varepsilon_{n} \log n
$$

for all k such that $k-m_{0}+1 \geq k_{0}$ or in other words, for all $n>n_{0}$.
Now the theorem follows by taking $C_{1}=\frac{1}{4} \mu_{1}^{2}(\delta-\varepsilon)^{2}$.

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